Topological Complexity and Motion Planning (Online) (20w5194)

Daniel C. Cohen (Louisiana State University),
Jesús González (Cinvestav),
Lucile Vandembroucq (Universidade do Minho)

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1 Overview of the Field

Topological complexity is a numerical homotopy invariant of a space introduced by Farber [3] with the aim to providing a topological measure of the complexity of the motion planning problem in robotics. This is a recently created area of research located within the burgeoning field of applied algebraic topology. With an originally applied motivation, this field has rapidly evolved with contributions from many theoretical topologists. More recently, applied flavored results have begun to re-emerge, and nowadays the subject is a vibrant and highly active and fruitful ground for scientific activity, ready to face the current needs of our technological society.

2 Presentation Highlights and Scientific Progress Made

The inaugural talk of the online workshop, "Topology of parametrised motion planning algorithms", was delivered by M. Farber, in joint work with D. Cohen and S. Weinberger. With this work, the authors established new directions of research in the subject, as the original concept for topological complexity (TC) was brought closer to the applied word by introducing "parameterised topological complexity". In the parametrised setting, a motion planning algorithm has high degree of universality and flexibility, and can function under a variety of external conditions (such as positions of the obstacles). In particular, by computing the parameterised topological complexity of obstacle-avoiding collision-free motion of many particles (robots) in 3-dimensional space, the authors showed that the parameterised topological complexity can be significantly higher than the standard (non-parametrised) invariant.

The workshop goal of connecting TC ideas to real life problems in robotics was also addressed through lectures delivered by D. Koditschek, V. Vasilopoulos, P. Gustafson and M. Kvalheim, in the first day of workshop activities. The starting point in this series of talks was the long tradition in robotics of deploying dynamical systems as reactive motion planners, where motion planning goals are encoded as attracting sets, and obstacles as repelling sets of vector fields arising from suitably constructed feedback laws [6]. This raised the prospects for a topologically informed notion of "closed loop" planning complexity [1], and whose contrast with Farber's "open loop" notion is of mathematical interest. Also in the applied side of the theory, A. Borat's lecture "A simplicial analog of homotopic distance" described a discretized model of the notion of distance for homotopy classes of maps introduced in [7]. This establishes a theoretical basis for computational implementations of homotopy distance and, in particular, of the several variants known for TC.

The second day of activities saw the introduction of Murillo-Wu's topological complexity of the work map associated to a robot system [8], which measures the complexity of any algorithm controlling, not just the motion of the configuration space of the given system, but also the task for which the system has been designed. From a purely topological point of view, this is a homotopy invariant of a map which generalizes Farber's TC. Then, in P. Pavesic's lecture "Two questions on TC", the author introduced the concept of a TC-regular space, a condition assuring that Farber's TC agrees with Iwase-Sakai's notion of monoidal TC [5], in order to clarify the relationship between works of Farber and of Dranishnikov on the TC of a one-point union of two spaces. Additionally, Pavesic made a thorough study of closed manifolds having small TC, thus extending work of Grant-Lupton-Oprea [4]. The lecture "Morita Invariance of Invariant Topological Complexity", by H. Colman (joint with Angel, Grant and Oprea), completed the second day of activities. In their work, the authors use the homotopy invariance of equivariant principal bundles to prove that the equivariant A-category of Clapp and Puppe is invariant under Morita equivalence. As a result, both the equivariant Lusternik-Schnirelmann category and the invariant topological complexity of a group action are shown to be invariant under Morita equivalence, thus yielding well defined notions for orbifolds.

Continuing with the group action viewpoint, the third day of activities featured S. Mescher's lecture "Spherical complexities and closed geodesics", where a new integer-valued homotopy invariant for topological spaces X is presented. The driving motivation is to get a Lusternik-Schnirelmann-type approach to counting critical orbits of G-invariant functions on subspaces of $C^0(S^n, X)$, where G is a closed subgroup of $O(n+1)$ acting on $C^0(S^n, X)$ by reparametrization. The resulting invariant, the n-th spherical complexity, is the sectional category of a fibration generalizing Farber's TC. Using sectional category weights of cohomology classes, the author described lower estimates for these invariants and, as an application, he outlined a way to derive new existence results for closed geodesics of Finsler metrics of positive flag curvature on spheres. Later, in his talk "Euler characteristics of exotic configuration spaces", Y. Baryshnikov explained how exponential generating functions for Euler characteristics of exotic configuration spaces have a remarkably simple representation in terms of the local geometry of the underlying spaces, while in his lecture "On topological complexity of hyperbolic groups", A. Dranishnikov discussed a recent proof of the equality $\mathrm{TC}(G) = 2\mathrm{cd}(G)$ for a nonabelian hyperbolic group G.

In the fourth and last day of activities, J. Oprea delivered the talk "Logarithmicity, the TC-generating function and right-angled Artin groups". Here, the known interplay between the combinatorics of a simplicial graph Γ and the topology of the classifying space for the right-angled Artin group determined by Γ is used to explain, and generalize to the higher TC realm, Dranishnikov's example [2] of a covering space having larger TC than its base space. Additionally, the existence of currently conjectured properties for the TC-generating function of a space X are connected by Oprea to the possibility of X satisfying logarithmicity of LS-category. Two final lectures by D. M. Davis and D. Recio-Mitter were given on Geodesic Complexity, a newly introduced variant of Farber's TC, which is based on motion planning along shortest paths. In Recio-Mitter's lecture (joint with D. M. Davis and M. Harrison), the authors introduce the total cut locus in order to show that geodesic complexity is sensitive to the metric and, in general, differs from Farber's TC. The authors also construct the first optimal motion planners on configuration spaces of graphs along shortest paths. On the other hand, in Davis' lecture, "Geodesic complexity of non-geodesic spaces", the author defines the notion of near geodesic between points where no geodesic exists, and use this to define geodesic complexity for non-geodesic spaces. Explicit near geodesics and geodesic complexity in a variety of cases are computed.

On each day of activities, lectures were complemented by virtual coffee breaks, providing participants with opportunities to interact in an informal setting. Several of these breaks featured lively discussions involving participants from both the pure and applied aspects of the area.

3 Outcome of the Meeting

The workshop was highly formative, with discussions of state-of-the-art problems in the subject, and with the participation of many of the leading experts in the area. Topological robotics is a newly and rapid growing research area and, through the activities of the workshop, new lines of research were clearly identified.

The workshop was originally scheduled as a 5-day in-person activity. But the current pandemic situation forced an online format. Yet, a very positive situation resulted from the change: The somewhat restricted (due to space restrictions) original participation of young researchers and students was fixed with the expanded

300-participants limit. As reported in the participant testimonials, the activities of the workshop impacted very positively on young researchers. In particular, a large number of graduate students were able to join the presentations and social activities in chat rooms. The impact on these future scientists will be an important positive consequence of the activities of the Topological Complexity and Motion Planning workshop.

In response to the extensive interest in the subject illustrated by the large number of workshop participants (over 150 confirmed), two of the organizers (DC and JG), in collaboration with J. Oprea (Cleveland State University), launched an ongoing online *Topological Complexity Seminar* not long after the workshop. This seminar, supported by the Applied Algebraic Topology Research Network (AATRN), featured three talks in Fall 2020, with five additional lectures already scheduled for Spring 2021. After they are delivered, lectures are posted and may be viewed on the AATRN YouTube Channel. Further information may be found at the seminar web page:

`https://sites.google.com/view/aatrn-tc-seminar/home`

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